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# The embeddability ordering of topological spaces <sup>☆</sup>

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Respectfully dedicated to Alexander “Shura” Arhangel’skiĭ on the occasion of his 65 th birthday

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## Abstract

For  $\mathcal{K}$  a set of topological spaces and  $X, Y \in \mathcal{K}$ , the notation  $X \subseteq_h Y$  means that  $X$  embeds homeomorphically into  $Y$ ; and  $X \sim Y$  means  $X \subseteq_h Y \subseteq_h X$ . With  $\tilde{X} := \{Y \in \mathcal{K} : X \sim Y\}$ , the equivalence relation  $\sim$  on  $\mathcal{K}$  induces a partial order  $\leq_h$  well-defined on  $\mathcal{K}/\sim$  as follows:  $\tilde{X} \leq_h \tilde{Y}$  if  $X \subseteq_h Y$ .

For posets  $(P, \leq_P)$  and  $(Q, \leq_Q)$ , the notation  $(P, \leq_P) \hookrightarrow (Q, \leq_Q)$  means: there is an injection  $h : P \rightarrow Q$  such that  $p_0 \leq_P p_1$  in  $P$  if and only if  $h(p_0) \leq_Q h(p_1)$  in  $Q$ . For  $\kappa$  an infinite cardinal, a poset  $(Q, \leq_Q)$  is a  $\kappa$ -universal poset if every poset  $(P, \leq_P)$  with  $|P| \leq \kappa$  satisfies  $(P, \leq_P) \hookrightarrow (Q, \leq_Q)$ .

The authors prove two theorems which improve and extend results from the extensive relevant literature.

**Theorem 2.2.** *There is a zero-dimensional Hausdorff space  $S$  with  $|S| = \kappa$  such that  $(\mathcal{P}(S)/\sim, \leq_h)$  is a  $\kappa$ -universal poset.*

**Theorem 3.1.** *There are a compact, connected Hausdorff space  $S$  and a set  $\mathcal{K}$  of  $(2^\kappa)$ -many compact, connected subspaces of  $S$  such that (a) the posets  $(\mathcal{P}(\kappa), \subseteq)$  and  $(\mathcal{K}/\sim, \leq_h)$  are isomorphic; and (b)  $(\mathcal{K}/\sim, \leq_h)$  is a  $\kappa$ -universal poset. Further, one may arrange  $|S| = w(S) = |X| = w(X) = \aleph_\kappa \cdot c$  for each  $X \in \mathcal{K}$ .*

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## 1. Introduction

### 1.1. Notation

The symbols  $\kappa$ ,  $\lambda$  and  $\mu$  denote cardinal numbers, and  $\xi$  and  $\eta$  are ordinals. For  $S$  a set we denote by  $\mathcal{P}(S)$  the power set of  $S$ , and when  $\kappa$  is a cardinal we write  $[S]^\kappa = \{A \in \mathcal{P}(S) : |A| = \kappa\}$ . When  $S$  is a space the elements of  $\mathcal{P}(S)$  are themselves spaces (with the inherited topology).

For  $\mathcal{K}$  a set of topological spaces and  $X, Y \in \mathcal{K}$ , the notation  $X \subseteq_h Y$  means that  $X$  is homeomorphic to a subspace of  $Y$ ; when  $X \subseteq_h Y \subseteq_h X$  we write  $X \sim Y$ , and we set  $\tilde{X} := \{Y \in \mathcal{K} : X \sim Y\}$ . The set  $\mathcal{K}/\sim$  of equivalence classes is partially ordered by the relation:  $\tilde{X} \leq_h \tilde{Y}$  if  $X \subseteq_h Y$ ; the resulting poset is denoted  $\tilde{\mathcal{K}} := (\mathcal{K}/\sim, \leq_h)$ . In particular, for a space  $S$  we have  $\tilde{\mathcal{P}(S)} := (\mathcal{P}(S)/\sim, \leq_h)$ .

For  $\kappa$  an infinite cardinal, a poset  $(Q, \leq_Q)$  is a  $\kappa$ -universal poset if every poset  $(P, \leq_P)$  with  $|P| \leq \kappa$  satisfies  $(P, \leq_P) \hookrightarrow (Q, \leq_Q)$ . We remark for emphasis that according to our usage a  $\kappa$ -universal poset  $(Q, \leq_Q)$  need not satisfy  $|Q| \leq \kappa$ .

### 1.2. Background

The literature in this corner of mathematics normally addresses two types of questions, as follows:

- (A) A space  $S$  is given, and one asks: which posets  $(P, \leq_P)$  satisfy  $(P, \leq_P) \hookrightarrow \tilde{\mathcal{P}(S)}$ ?
- (B) A poset  $(P, \leq_P)$  is given, and one asks: Is there a space  $S$  such that  $(P, \leq_P) \hookrightarrow \tilde{\mathcal{P}(S)}$ ?

[For every  $(P, \leq_P)$  the answer to (B) is “Yes”, so attention is typically directed to more subtle questions, such as these two.]

- (i) What is  $\min\{|S| : (P, \leq_P) \hookrightarrow \tilde{\mathcal{P}(S)}\}$ ?
- (ii) Can  $S$ , and/or the elements of  $P \hookrightarrow \mathcal{P}(S)$ , be chosen with special pre-assigned topological properties—e.g., connected, compact, ...?

### 1.3. Discussion

(a) In the spirit of (A), with  $S = \mathbb{R}$ , we cite a pleasing recent result of McCluskey et al. [14]:  $(\mathcal{P}(\mathfrak{c}), \subseteq) \hookrightarrow \tilde{\mathcal{P}(\mathbb{R})}$ . It follows in particular, since  $(\mathcal{P}(\mathfrak{c}), \subseteq)$  is easily seen to contain an antichain of cardinality  $2^{\mathfrak{c}}$ , that  $\mathbb{R}$  contains a set of  $2^{\mathfrak{c}}$ -many subspaces, no one of which embeds homeomorphically into another; that specific statement dates back to 1926 (cf. Kuratowski [11]). The relation  $(\mathcal{P}(\mathfrak{c}), \subseteq) \hookrightarrow \tilde{\mathcal{P}(\mathbb{R})}$  does not tell the full story

about  $\widetilde{\mathcal{P}(\mathbb{R})}$ , however, since  $(\mathcal{P}(c), \subseteq)$  contains no well-ordered set of cardinality  $c^+$  while  $\widetilde{\mathcal{P}(\mathbb{R})}$  does contain such a set [12].

(b) Similarly in the spirit of (A), one of us has recently achieved a poset-theoretic characterization of the poset  $\widetilde{\mathcal{P}(\mathbb{Q})}$  [7].

(c) In the direction of (B), McCluskey and McMaster [13] have shown that for  $\kappa \geq \omega$  there is a space  $S$  with  $|S| = \kappa$  such that  $\widetilde{\mathcal{P}(S)}$  is a  $\kappa$ -universal poset. The spaces  $S$  constructed in [13] have the advantage that they contain large connected subspaces; but they are neither  $T_1$ -spaces nor regular, hence they are not Tychonoff spaces. In contrast, our argument in Theorem 2.2 below furnishes for each  $\kappa \geq \omega$  a Tychonoff space  $S = S_\kappa$  of cardinality  $\kappa$  such that  $\widetilde{\mathcal{P}(S)}$  is a  $\kappa$ -universal poset.

(d) We note that Trřková [17] has established a strong version of the statement that the class of Tychonoff spaces contains antichains of arbitrarily large cardinality. Indeed, according to [17], there is a proper class  $\mathcal{K}$  of paracompact Hausdorff spaces such that

- (i) each  $X$  in  $\mathcal{K}$  is *strongly rigid* in the sense that each continuous  $f: X \rightarrow X$  satisfies either  $f = \text{id}_X$  or  $f$  is a constant function (i.e.,  $|f[X]| = 1$ ); and
- (ii) the class  $\mathcal{K}$  is *strongly bi-rigid* in the sense that for distinct  $X, Y$  in  $\mathcal{K}$  each continuous  $f: X \rightarrow Y$  satisfies  $|f[X]| = 1$ .

(e) Though it plays no role in this paper, we cannot resist citing this theorem of Schroeder–Bernstein type established in 1924 by Banach [1]: If spaces  $X$  and  $Y$  satisfy  $X \sim Y$ , then there are homeomorphic subsets  $A$  and  $B$  of  $X$  and  $Y$ , respectively, such that  $X \setminus A$  and  $Y \setminus B$  are also homeomorphic.

(f) It has been noted over the years by many workers that each poset  $(P, \leq_P)$  embeds into the poset  $(\mathcal{P}(P), \subseteq)$  (in symbols:  $(P, \leq_P) \hookrightarrow (\mathcal{P}(P), \subseteq)$ ). Indeed the map  $h: P \rightarrow \mathcal{P}(P)$  given by  $h(x) = \{y \in P: y \leq_P x\}$  is an embedding as required. We use this fact frequently in what follows, without additional warning.

(g) It follows from the cited relation  $(P, \leq_P) \hookrightarrow (\mathcal{P}(P), \subseteq)$  that for  $\kappa \geq \omega$  the poset  $(\mathcal{P}(\kappa), \subseteq)$  is a  $\kappa$ -universal poset (of cardinality  $2^\kappa$ ).

## 2. A small, zero-dimensional Tychonoff space $S$ with $\widetilde{\mathcal{P}(S)}$ $\kappa$ -universal

For  $\omega \leq \alpha \leq \kappa$ , a family  $\mathcal{A} \subseteq \mathcal{P}(\kappa)$  is said to be  $\alpha$ -almost disjoint if distinct elements  $A, B$  of  $\mathcal{A}$  satisfy  $|A \cap B| < \alpha$ .

When  $\kappa$  is an infinite cardinal, the symbol  $\beta(\kappa)$  denotes the Stone–Ćech compactification of the (discrete) space  $\kappa$ . An ultrafilter  $p \in \beta(\kappa)$  is *uniform* over  $\kappa$  if  $p \subseteq [\kappa]^\kappa$ ; the set of ultrafilters uniform over  $\kappa$  is denoted  $U(\kappa)$ . When  $A \subseteq \kappa$  we have  $\beta(A) = \text{cl}_{\beta(\kappa)} A$  and we write  $U(A) := U(\kappa) \cap \beta(A)$ . (According to this convention we have  $U(A) = \emptyset$ , if  $|A| < \kappa$ .) Given  $A, B \subseteq \kappa$  and  $f: A \rightarrow B \subseteq \beta(\kappa)$  we denote by  $\bar{f}$  that continuous function  $\bar{f}: \beta(A) \rightarrow \beta(B)$  such that  $\bar{f}|_A = f$ .

**Lemma 2.1.** *Let  $\omega \leq \kappa \leq \lambda$  and suppose there is a  $\kappa$ -almost disjoint family  $\mathcal{A} \subseteq [\kappa]^\kappa$  such that  $|\mathcal{A}| = \lambda$ . Then there is a set  $S \subseteq \beta(\kappa)$  such that*

- (a)  $|S| = \lambda$ , and  
 (b)  $\widetilde{\mathcal{P}(S)}$  is a  $\kappa$ -universal poset.

**Proof.** For  $A \in \mathcal{A}$  we choose  $p_A \in U(A)$  such that if  $\{A, B\} \in [\mathcal{A}]^2$  there is no injection  $f: A \rightarrow B \subseteq \beta(B)$  such that  $\bar{f}(p_A) = p_B$ . [The availability of such a set  $\{p_A: A \in \mathcal{A}\}$  follows from well-known facts which are elucidated in detail in [2]. We content ourselves here with a brief outline. First, adopting terminology of Rudin [16] and Frolík [4,5] to the effect that the *type*  $\tau(p)$  of an ultrafilter  $p \in \beta(\kappa)$  is the set

$$\tau(p) = \{q \in \beta(\kappa): \text{there is a permutation } f: \kappa \rightarrow \kappa \text{ such that } \bar{f}(p) = q\},$$

one has  $|\tau(p)| \leq \kappa^\kappa = 2^\kappa$  (with equality easily shown). From  $|U(\kappa)| = 2^{2^\kappa}$  (cf. [2, (7.8)]) it then follows that the set  $T := \{\tau(p): p \in U(\kappa)\}$  of “uniform types” satisfies  $|T| = 2^{2^\kappa}$ . It is known, further, that if  $A \in [\kappa]^\kappa$  and  $p \in U(\kappa)$  then  $U(A) \cap \tau(p) \neq \emptyset$  (cf. [2]). It is enough, then, to choose an injection  $i: \mathcal{A} \rightarrow T$  and then for  $A \in \mathcal{A}$  to choose  $p_A \in U(A)$  such that  $\tau(p_A) \in i(A)$ .] We claim that for  $A \in \mathcal{A}$  the relation  $\kappa \cup \{p_A\} \subseteq_h \kappa \cup \{p_B: B \in \mathcal{A} \setminus \{A\}\}$  fails. Indeed, suppose that  $h: \kappa \cup \{p_A\} \rightarrow \kappa \cup \{p_B: B \in \mathcal{A} \setminus \{A\}\}$  is an embedding. Then  $h(p_A) \notin \kappa$  (since  $p_A$  is not isolated in  $\kappa \cup \{p_A\}$ ), so  $h(p_A) = p_B$  for some  $B \in \mathcal{A} \setminus \{A\}$ . The set  $\text{cl}_{\beta(\kappa)} B$  is an open neighborhood in  $\beta(\kappa)$  of  $p_B$ , and  $\text{cl}_{\beta(\kappa)} B \cap \{p_A: A \in \mathcal{A}\} = \{p_B\}$  since the family  $\mathcal{A}$  is  $\kappa$ -almost disjoint and the ultrafilters  $p_A$  are uniform. Since  $p_B \in \text{cl}_{\beta(\kappa)}(h[\kappa] \cap B)$  and  $p_B$  is uniform, we have  $|h[\kappa] \cap B| = \kappa$ , so  $h[\kappa] \cap B$  can be partitioned into two complementary sets  $U$  and  $V$ , each of cardinality  $\kappa$ , say with  $U \in p_B$  and  $V \notin p_B$ . Then any permutation  $h': \kappa \rightarrow \kappa$  such that  $h'[h^{-1}[U]] = h[h^{-1}[U]]$  and  $h'[\kappa \setminus h^{-1}[U]] = \kappa \setminus U$  satisfies  $\bar{h}'(p_A) = \bar{h}(p_A) = p_B$  and hence witnesses the relation  $\tau(p_A) = \tau(p_B)$ , a contradiction. The claim is established.

We define  $S := \kappa \cup \{p_A: A \in \mathcal{A}\}$ , and for  $\mathcal{X} \subseteq \mathcal{A}$  we set  $S_{\mathcal{X}} := \kappa \cup \{p_A: A \in \mathcal{X}\}$ . Clearly for  $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{A}$  we have  $S_{\mathcal{X}} \subseteq S_{\mathcal{Y}}$  and hence  $\widetilde{S_{\mathcal{X}}} \leq_h \widetilde{S_{\mathcal{Y}}}$ . If  $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}$  and  $\mathcal{X} \subseteq \mathcal{Y}$  fails (say  $A \in \mathcal{X} \setminus \mathcal{Y}$ ) then  $\mathcal{Y} \subseteq \kappa \cup \{p_B: B \in \mathcal{A} \setminus \{A\}\}$ , so according to the claim established above, the relation  $\kappa \cup \{p_A\} \subseteq_h S_{\mathcal{Y}}$  fails and hence  $\widetilde{S_{\mathcal{X}}} \not\leq_h \widetilde{S_{\mathcal{Y}}}$  fails. The map  $\mathcal{X} \rightarrow S_{\mathcal{X}}$  thus establishes the relation  $(\mathcal{P}(\mathcal{A}), \subseteq) \hookrightarrow \widetilde{\mathcal{P}(S)}$ , and for every poset  $(P, \leq_P)$  with  $|P| \leq \lambda$  we have

$$(P, \leq_P) \hookrightarrow (\mathcal{P}(P), \subseteq) = (\mathcal{P}(\mathcal{A}), \subseteq) \hookrightarrow \widetilde{\mathcal{P}(S)},$$

as required.  $\square$

**Theorem 2.2.** Let  $\kappa \geq \omega$ . There are zero-dimensional Tychonoff spaces  $S_0$  and  $S_1$  such that  $|S_0| = \kappa$ ,  $|S_1| = \kappa^+$ ,  $w(S_i) \leq 2^\kappa$ , and

- (0)  $\widetilde{\mathcal{P}(S_0)}$  is a  $\kappa$ -universal poset; and  
 (1)  $\widetilde{\mathcal{P}(S_1)}$  is a  $\kappa^+$ -universal poset.

If in addition  $\kappa = 2^{<\kappa} := \sum_{\mu < \kappa} 2^\mu$ , then there is a zero-dimensional Tychonoff space  $S_2$  such that  $|S_2| = 2^\kappa$ ,  $w(S_i) = 2^\kappa$ , and

- (2)  $\widetilde{\mathcal{P}(S_2)}$  is a  $2^\kappa$ -universal poset.

**Proof.** In each case (i) ( $0 \leq i \leq 2$ ) we take  $S_i = \kappa \cup \{p_A : A \in \mathcal{A}_i\} \subseteq \beta(\kappa)$  as in Lemma 2.1, with  $|\mathcal{A}_0| = \lambda = \lambda_0 := \kappa$ ,  $|\mathcal{A}_1| = \lambda = \lambda_1 := \kappa^+$ , and  $|\mathcal{A}_2| = \lambda = \lambda_2 := 2^\kappa$ . That such a  $\kappa$ -almost disjoint family  $\mathcal{A}_0 \subseteq \mathcal{P}(\kappa)$  exists is obvious; that such  $\mathcal{A}_1$  exists is well known (see for example [18, p. 2] or [2, 12.6]); the existence of such  $\mathcal{A}_2$  (when  $\kappa = 2^{<\kappa}$ ) is a theorem of Tarski (see [2, (12.2)]).  $\square$

**Remark 2.3.** *En route* to their construction mentioned above of a connected, non-Tychonoff, countable space  $S$  such that  $\widetilde{\mathcal{P}(S)}$  is an  $\omega$ -universal poset, the authors of [13] demonstrate (by a direct, inductive argument) for  $\kappa \geq \omega$  the existence of  $2^{2^\kappa}$ -many distinct ultrafilter types over  $\kappa$ .

**Discussion 2.4.** The spaces  $S_0$  of Theorem 2.2(0) for which  $\widetilde{\mathcal{P}(S_0)}$  is a  $\kappa$ -universal poset satisfy  $|\widetilde{\mathcal{P}(S_0)}| = 2^{|S_0|} = 2^\kappa$ . The present authors do not know whether, given suitably restricted  $\kappa \geq \omega$ , there is a space  $S$  with  $|S| < \kappa$  such that  $\widetilde{\mathcal{P}(S)}$  is a  $\kappa$ -universal poset. (Naively, since  $|\mathcal{P}(\log(\kappa))| \geq \kappa$ ,  $|S| = \log(\kappa) < \kappa$  might occur—but we know of no such instance.) Specifically, we ask:

(A) Given  $\kappa \geq \omega$ , what is  $\min\{|S| : \widetilde{\mathcal{P}(S)} \text{ is a } \kappa\text{-universal poset}\}$ ?

In this context, the following even more fundamental question arises.

(B) Given  $\kappa \geq \omega$ , what is the least cardinal of a  $\kappa$ -universal poset?

Let us denote that cardinal by the symbol  $m(\kappa)$ . It is known that  $m(\omega) = \omega$ ; indeed, the witnessing countable poset  $S$  can be chosen in addition to be  $\omega$ -homogeneous in the sense that every isomorphism between finite subposets extends to an automorphism of  $S$ . (More generally, for every Jónsson Class  $\mathbb{K}$  and for  $\omega \leq \alpha \leq \kappa = \kappa^{<\alpha}$  there is an  $\alpha$ -homogeneous,  $\alpha$ -universal  $S \in \mathbb{K}$  such that  $|S| = \kappa$ ; a  $\kappa$ -homogeneous,  $\kappa$ -universal  $S \in \mathbb{K}$  with  $|S| = \kappa$  exists if and only if  $\kappa = \kappa^{<\kappa}$ . See [15] or [2, §4] for detailed proofs in a broad context, and see [8, §6] for a treatment in modern language of the countable case.) In any case, the statement  $\kappa \geq \omega \Rightarrow m(\kappa) = \kappa$  is not a theorem of ZFC: Arguing in ZFC, Kojman and Shelah [10] show that  $m(\kappa) > \kappa$  for many singular cardinals  $\kappa$ , and for regular  $\kappa$  such that  $\omega^+ < \kappa < \mathfrak{c}$ . (The treatment in [10] focuses on linearly ordered sets rather than posets, but  $\kappa$ -universal structures of cardinality  $\kappa$  exist in those two classes for exactly the same  $\kappa$ .)

For classic studies which stimulated subsequent work on  $\kappa$ -universal posets, see Dushnik and Miller [3] and Johnston [9].

**Remark 2.5.** The referee has observed, in effect, that Question 2.4(A) invokes a spectrum of questions: Given  $\kappa \geq \omega$  and classes  $\mathbb{P}$  and  $\mathbb{Q}$  of spaces, is there a space  $S \in \mathbb{P}$  such that each poset  $(P, \leq_P)$  with  $|P| \leq \kappa$  admits an embedding  $h : (P, \leq_P) \hookrightarrow \widetilde{\mathcal{P}(S)}$  such that each equivalence class  $\widetilde{h(x)} \in \widetilde{\mathcal{P}(S)}$  contains a space in  $\mathbb{Q}$ ? When such  $S$  exists, what is the smallest achievable cardinality?

### 3. A large, compact, connected space $S$ with $\widetilde{\mathcal{P}(S)}$ $\kappa$ -universal

**Theorem 3.1.** *Let  $\kappa \geq \omega$ . There are a compact, connected, Tychonoff space  $S$  and a family  $\mathcal{K}$  of compact, connected subspaces of  $S$  such that the posets  $(\mathcal{P}(\kappa), \subseteq)$  and  $\widetilde{\mathcal{K}}$  are isomorphic. One may arrange that  $|S| = w(S) = |X| = w(X) = \aleph_\kappa \cdot \mathfrak{c}$  for each  $X \in \mathcal{K}$ .*

**Proof.** Let  $L := (\aleph_\kappa \times [0, 1], \text{lex}) \cup \{\aleph_\kappa\}$  be the one-point compactification of “the long ray of length  $\aleph_\kappa$ ”, and for each successor cardinal  $\lambda = \aleph_{\xi+1}$  ( $\xi < \kappa$ ) let  $L(\lambda)$  denote the space  $L$  with the interval  $I_\lambda := [0, 1]$  attached at the point  $(\lambda, 0)$ . (More precisely:  $L(\lambda)$  is the disjoint union of  $L$  with  $I_\lambda$ , with the points  $(\lambda, 0) \in L$  and  $0 \in I_\lambda$  identified.)

We claim that for  $\lambda = \aleph_{\xi+1}$  and  $\mu = \aleph_{\eta+1}$  with  $\{\xi, \eta\} \in [\kappa]^2$  the relation  $L(\lambda) \subseteq_h L(\mu)$  fails. The point  $(\lambda, 0)$  of  $L(\lambda)$  has the property that

- (i) the local weight of  $L(\lambda)$  at  $(\lambda, 0)$  satisfies  $\chi(L(\lambda), (\lambda, 0)) = \lambda = \aleph_{\xi+1}$ , and
- (ii) the space  $L(\lambda) \setminus \{(\lambda, 0)\}$  has three connected components.

No connected subspace of  $L(\mu)$  contains a point with local weight  $\lambda$  whose deletion yields a space with three components, so there is no injective homeomorphism from  $L(\lambda)$  into  $L(\mu)$ . The claim is established.

For  $\mathcal{X} \subseteq \kappa$  we set  $S_{\mathcal{X}} := L \cup \bigcup \{L(\lambda) : \lambda = \aleph_{\xi+1}, \xi < \kappa, \xi \in \mathcal{X}\}$ , with all “base points”  $(0, 0) \in L$  and  $(0, 0) \in L(\lambda)$  identified. (Thus,  $L_\emptyset = L$ .)  $L(\lambda)$  is topologized so that each non-base point of  $L$  and each non-base point of  $L(\lambda)$  has the same basic open neighborhood family as originally, while basic neighborhoods of the new base point  $(0, 0)$  are given by restriction in finitely many rays; that is, a set  $U$  containing  $(0, 0)$  is open if

- (i)  $U \cap L$  is open in the original topology of  $L$ ;
- (ii) for  $\lambda = \aleph_{\xi+1}$ ,  $\xi \in \mathcal{X}$ ,  $U \cap L(\lambda)$  is open in the original topology of  $L(\lambda)$ ; and
- (iii) the inclusion  $L(\lambda) \subseteq U$  fails for at most finitely many  $\lambda$ .

It is clear that each  $S_{\mathcal{X}}$  is a compact, connected, Hausdorff space.

We define  $\mathcal{K} := \{S_{\mathcal{X}} : \mathcal{X} \subseteq \kappa\}$ . Clearly for  $\mathcal{X} \subseteq \mathcal{Y} \subseteq \kappa$  we have  $S_{\mathcal{X}} \subseteq S_{\mathcal{Y}}$  and hence  $\widetilde{S_{\mathcal{X}}} \leq_h \widetilde{S_{\mathcal{Y}}}$ . If  $\mathcal{X}, \mathcal{Y} \subseteq \kappa$  and  $\mathcal{X} \subseteq \mathcal{Y}$  fails (say  $\xi \in \mathcal{X} \setminus \mathcal{Y}$ ) then according to the claim just established the relation  $L(\aleph_{\xi+1}) \subseteq_h S_{\mathcal{Y}}$  fails and hence  $\widetilde{S_{\mathcal{X}}} \leq_h \widetilde{S_{\mathcal{Y}}}$  fails. The map  $\mathcal{X} \rightarrow S_{\mathcal{X}}$  thus establishes a poset isomorphism from  $(\mathcal{P}(\kappa), \subseteq)$  onto  $\widetilde{\mathcal{K}}$ , as required.  $\square$

**Remark 3.2.** From earlier observations it follows for  $S$  as in Theorem 3.1 that  $\widetilde{\mathcal{P}(S)}$  is  $\kappa$ -universal. The principal point here is that for every poset  $(P, \leq_P)$  with  $|P| \leq \kappa$  the witnessing embedding  $h : (P, \leq_P) \hookrightarrow \widetilde{\mathcal{K}} \subseteq \widetilde{\mathcal{P}(S)}$  can be achieved so that  $h(x)$  has a compact and connected representative for each  $x \in P$ . For all we know, the cardinal number  $|S| = \aleph_\kappa \cdot \mathfrak{c}$  in the proof just given may be larger than necessary. In parallel with the questions posed in Section 2, we ask:

**Question 3.3.** Let  $\kappa \geq \omega$ . What is the minimum cardinality of a compact, connected, Tychonoff space  $S$  such that every poset  $(P, \leq_P)$  with  $|P| \leq \kappa$  admits an embedding

$h : (P, \leq_P) \hookrightarrow \widehat{\mathcal{P}(S)}$  with  $h(x)$  containing a compact and connected representative for each  $x \in P$ ?

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